# A Unified Discussion of High-$Q$ Waveguide Filter Design Theory*

HENRY J. RIBLET†

*Summary*—For the general design of conventional, high-$Q$, direct-coupled waveguide filters to be based on the frequency behavior of a classical ladder network prototype, it is necessary and sufficient that the reflecting elements of the filter be replaceable by admittance inverters and that the lengths of transmission line be replaceable by resonant elements. The error due to the latter assumption is of the order of twice the square of the percentage bandwidth measured in guide wavelengths, and the classical synthesis problem is a limiting case of a solvable transmission line problem. In this limit, an exact equivalence is established between the design of a direct-coupled filter and the design of a quarter-wave-coupled filter based on the same ladder network prototype. Design formulas for equal ripple and maximally flat performance are given for the VSWR's of the reflecting elements in terms of dimensionless quantities. Detailed comparison of previous formulas is made.

## Introduction

THIS PAPER is concerned with the general design of filters which consist of a cascade of large, lossless, similar, reflecting elements, often shunt inductances, spaced in a regular manner on a uniform waveguide. Lawson and Fano[1] have given general synthesis procedures for two types of filters distinguished as "quarter-wave-coupled" and "direct-coupled." For both types of filters, they give explicit synthesis procedures based on the use of a ladder network prototype having a prescribed insertion loss function. Southworth[2] has given the design parameters for direct-coupled, maximally flat filters without, however, any supporting synthesis procedure. Mumford[3] has given the design parameters for quarter-wave-coupled, maximally flat filters, and has improved on the approximation used by Lawson and Fano for the interconnecting, quarter wavelength of waveguide. He has used the synthesis procedure proposed by Lawson and Fano. Recently, Cohn[4] has given design parameters for direct-coupled filters, for equal ripple, and for maximally flat response. He has employed a ladder network prototype explicitly and has used a frequency transformation which improves the approximation for shunt inductances.

A general synthesis procedure for the design of direct-coupled filters has been given by Riblet[5] but since it is not based on the use of a ladder network prototype, it is not within the scope of this paper.

The multiplicity of papers in the literature which are concerned with the design of direct-coupled filters (Fig. 1), is a source of confusion to the design engineer. Lawson and Fano,[1] Cohn,[4] and presumably Southworth,[2] base their assumptions on the same ladder network prototype (Fig. 2). In spite of this similarity, Cohn has indicated how different results are obtained in each of these papers. Actually the situation is not difficult to understand when one considers the number of approximations involved in the synthesis procedure and realizes that little effort has been made to justify any of them rigorously.

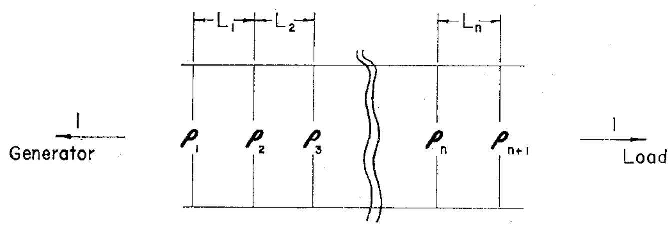


Fig. 1—Schematic of a direct-coupled filter.

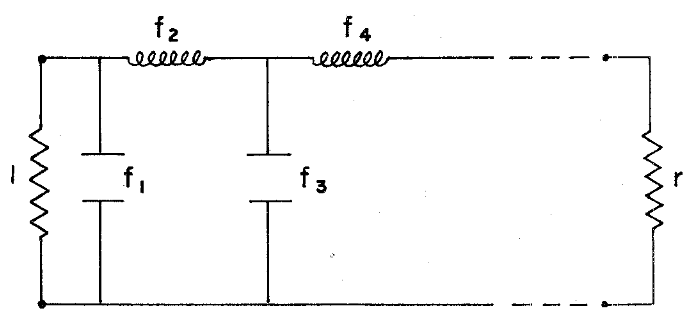


Fig. 2—Ladder network prototype.

The fact that the differences observed by Cohn are minor suggests that the methods used are the same in principle. It is the object of this paper to show that this is the case and to unify the whole problem by putting the approximations required for the existence of a general synthesis procedure, based on a ladder network prototype, on a formal basis. By way of justification it will be indicated how the approximate synthesis is a limiting

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† Microwave Development Lab., Wellesley, Mass.

[1] A. W. Lawson and R. M. Fano, "The design of microwave filters," in "Microwave Transmissions Circuits," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 9, pp. 661–706; 1948.

[2] G. C. Southworth, "Principles and Applications of Waveguide Transmission," D. Van Nostrand Co., Inc., New York, N. Y., pp. 297–298; 1950.

[3] W. W. Mumford, "Maximally-flat filters in waveguide," *Bell Sys. Tech. J.*, vol. 27, pp. 684–713; October, 1948.

[4] S. B. Cohn, "Direct-coupled resonator filters," Proc. IRE, vol. 45, pp. 187–196; February, 1957.

[5] H. J. Riblet, "Synthesis of narrow-band direct-coupled filters," Proc. IRE, vol. 40, pp. 1219–1223; October, 1952.

form of an exact transmission-line synthesis. As by-products, we obtain 1) a quantitative estimate of the error involved in the ladder network approximation, 2) slightly more concise formulas than previously given, and 3) a demonstration of an exact equivalence between direct-coupled filters and quarter-wave-coupled filters using Mumford's approximation for the quarter wavelength of line. Finally, the general synthesis given by Lawson and Fano[1] is derived, and it is shown how our formulas for the case of direct-coupled filters may be obtained as special cases from their formulas.

## Formal Theory

This section considers the consequences of the following two simplifying assumptions.

1) The admittance transformation of the reflecting elements, which relates the input voltage and current, $v_i$ and $i_i$ to the output voltage and current, $v_0$ and $i_0$, may be written

$$\begin{aligned} i_i &= 0\cdot i_0 + j\sqrt{\rho}v_0 \\ v_i &= \frac{j}{\sqrt{\rho}}\cdot i_0 + 0\cdot v_0, \end{aligned} \tag{1}$$

where $\rho$[6] is a positive real constant, to be called the inversion factor.

2) The admittance transformation of the interconnecting half wavelengths of waveguide, may be written

$$\begin{aligned} i_i &= 1\cdot i_0 + 0\cdot v_0 \\ v_i &= p\cdot i_0 + 1\cdot v_0. \end{aligned} \tag{2}$$

where $p = j \sin \theta$ with $\theta = 2\pi l/\lambda_g$.

Now the interest in 1) and 2) arises from the fact that they are, as will be shown in a very broad sense, necessary and sufficient for the existence of a general synthesis procedure for direct-coupled filters based on a ladder-network prototype. Accordingly, differences in previous results[1,2,4] must arise from the approximations used in relating $\rho$ and $p$ to reactance and frequency.

How this can be demonstrated is outlined in Appendix I. First, the possibility of connecting the reflecting elements with transmission line sections having the admittance transformation,

$$\begin{aligned} i_i &= \cos\theta i_0 + j\sin\theta v_0 \\ v_i &= j\sin\theta i_0 + \cos\theta v_0 \end{aligned} \tag{3}$$

is considered and then excluded when it is found that, in general, the input admittance of such a cascade cannot have the form required by ladder network prototypes. (Note that the coefficients of (1)–(3) are transfer matrices[7] rearranged in DCBA form.) When (3) is replaced by (2) such a synthesis can be carried out if, and only if, the general circuit parameters of the reflecting elements, assumed to be analytic in $p$, simplify to the form in (1). Eq. (2) is justified further in Appendix II, where it is shown that if the admittance inverters, designed on the basis of a ladder network prototype, are equally spaced on a uniform transmission line, and and if the inversion factors are allowed to become arbitrarily large so as to preserve the form of the insertion loss function, though not its bandwidth, then $\cos\theta$ tends to unity and the upper-corner $\sin\theta$ tends to zero, and (2) is exact. Of course, the validity of (1) remains to be established in each application.

To show that a direct-coupled filter consisting of a cascade of alternate (1) and (2) matrices corresponds to every ladder network prototype readily follows with the help of the equalities,

$$f_i\begin{pmatrix}1 & f_ip\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & f_i\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ p & 1\end{pmatrix}\begin{pmatrix}0 & f_i\\ 1 & 0\end{pmatrix} \tag{4}$$

and

$$\begin{pmatrix}0 & f_if_k\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & f_i\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & f_k\\ 1 & 0\end{pmatrix}. \tag{5}$$

Consider a ladder network prototype which can be written in matrix form as

$$\begin{pmatrix}1 & f_1p\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & f_2p\\ 0 & 1\end{pmatrix}\cdots\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & f_np\\ 1 & 0\end{pmatrix}, \tag{6}$$

since the first and alternate matrices are the admittance transformations of shunt resonant elements and the remaining matrices are the admittance inverters that correspond to changing from shunt to series. Applying (4)[8] to the resonant elements of (6) and eliminating the unity admittance inverters by means of (5) results in the matrix product,

$$\begin{pmatrix}0 & f_1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ p & 1\end{pmatrix}\begin{pmatrix}0 & f_1f_2\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ p & 1\end{pmatrix}\cdots\begin{pmatrix}0 & f_n\\ 1 & 0\end{pmatrix}, \tag{7}$$

which is of the required form if

$$\rho_1 = f_1,\ \rho_2 = f_1\cdot f_2,\ \cdots,\ \rho_n = f_n\cdot f_{n-1},\ \rho_{n+1} = f_n. \tag{8}$$

The converse is not generally true, but if we terminate the direct-coupled filter, we can always find a terminated ladder network prototype with the same input admittance function. Accordingly, it is convenient to carry through a general synthesis procedure for the determination of the $\rho$'s without requiring the $f$'s. We now consider the admittances seen at various points of the network assuming a known terminating admittance. If $Y_i = N_i/D_i$, where $N_i$ and $D_i$ are polynomials in $p$, is the admittance preceding

$$\begin{pmatrix}0 & \rho_i\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ p & 1\end{pmatrix}, \tag{9}$$

[6] $\rho$ is the coupling reactance, $X^{-2}$, of Lawson and Fano, *op. cit.* We have chosen to denote it by $\rho$ to emphasize that it is, for $\rho \geq 1$, numerically equal to VSWR of the reflecting element.

[7] E. A. Guillemin, "Communication Networks," John Wiley and Sons, Inc., New York, N. Y., vol. 2, pp. 144–152; 1935.

[8] Any common multiplicative factors may be ignored because they cancel out of numerator and denominator once a terminating admittance is selected and the input admittance is determined.

and $Y_{i-1}=N_{i-1}/D_{i-1}$ is the admittance into which it is transformed by (9), then

$$N_{i-1} = \rho_i p N_i + \rho_i D_i; \qquad D_{i-1} = N_i. \tag{10}$$

If the admittance, $Y_{n+1}$, terminating the cascade is a constant, then $Y_n$ is a first degree polynomial in $p$. In particular, the degree of $N_n$ is one higher than the degree of $D_n$. This is true for all admittances looking into the admittance inverters. For, if it is true for $N_i$ and $D_i$ in (10), it is certainly true for $N_{i-1}$ and $D_{i-1}$. Thus the input admittance of a cascade of the form of (7) is a rational function of $p$ whose denominator is of degree $n$ in $p$ while its numerator is of degree $n-1$. It is, of course, positive real in the sense of Brune.[9] Of course the input impedance of the network can be determined from the given insertion loss function by the method of Darlington.[10] Then the $\rho_i$ associated with $Y_{i-1}$ can always be determined by dividing the coefficient of the highest power of $p$ in its numerator by the coefficient of the highest power of $p$ in the denominator. This follows from (10). We also see that $Y_i$ equals the denominator, $D_{i-1}$, of $Y_{i-1}$ divided by the remainder when $N_{i-1}/\rho_i$ is divided by $D_{i-1}$.

This procedure can be summarized in a continued fraction expansion for the input admittance, $Y_0$, of the cascade terminated in an admittance, $R$.

$$Y_0 = \rho_1 p + \cfrac{\rho_1}{\rho_2 p + \cfrac{\rho_2}{\ddots \; \rho_n p + \cfrac{\rho_n}{(\rho_{n+1})/R}}} \tag{11}$$

The use of the continued fraction expansion simplifies certain manipulations once it is clear that the value of $Y_0$ is unchanged when one multiplies the quantities directly above and below any of the fraction bars by the same quantity. Thus in (10), we may replace the second $\rho_1$ by 1, if we divide both $\rho_2$'s below it by $\rho_1$. Thus $Y_0$ can be written

$$Y_0 = \rho_1 p + \cfrac{1}{(\rho_2/\rho_1) p + \cfrac{1}{\left(\dfrac{\rho_1\rho_3}{\rho_2}\right) p + \cfrac{1}{\ddots \; \left(\dfrac{\rho_n\rho_{n-2}\cdots}{\rho_{n-1}\rho_{n-3}\cdots}\right) p + \dfrac{\rho_n\rho_{n-2}\cdots R}{\rho_{n+1}\rho_{n-1}\cdots}}}} \tag{12}$$

Thus, for every terminated direct-coupled filter there is a corresponding terminated ladder network prototype having the same input impedance, although the terminations are not in general the same. This is precisely the equivalence used by Lawson and Fano[1] to give a general synthesis procedure for direct-coupled filters. This discussion, however, avoids the troublesome reactances, $X_S$ and $X_L$ which were invoked by them.

In general, the values of $p$ which are of practical interest are small, whereas the formulas due to Bennett[11] and Belevitch (modified by Orchard)[12] for the element values, $f_i$, of the ladder network prototypes yielding maximally flat and Tchebycheff performance assume that the value of the frequency variable $\omega$ is equal to unity at the edges of the pass band. Accordingly, we consider the effect of replacing $p$ in (11) by $tp$. We then find that the form of the continued fraction can be preserved, if we multiply numerator and denominator of each fraction bar by $t$. The resulting continued fraction is

$$Y_0 = \rho_1 t p + \cfrac{\rho_1 t}{\rho_2 t^2 p + \cfrac{\rho_2 t^2}{\ddots \; \rho_n t^2 p + \cfrac{\rho_n t^2}{(\rho_{n+1})r/R}}} \tag{13}$$

Thus a narrowing of the frequency scale by a factor of $1/t$ is accomplished by multiplying the inversion factors at each end of the filter by $t$ and all of the other inversion factors by $t^2$.

For narrow-band filters, the dimensional tolerances are relaxed by replacing some or all of the half wavelengths of waveguide by sections an integral number of half wavelengths long. Increasing the length of a filter in this way also increases its peak power handling ability. In fact, it is readily shown, by the methods of this section, that for fixed band-pass characteristics the maximum voltage in a given cavity section varies

[9] O. Brune, "Synthesis of a finite two-terminal network whose driving point impedance is a prescribed function of frequency," *J. Math. Phys.*, vol. 10, pp. 191–236; October, 1931.

[10] S. Darlington, "Synthesis of reactance 4-poles," *J. Math Phys.*, vol. 18, pp. 257–353; September, 1939

[11] W. R. Bennett, U. S. Patent No. 1,849,656; March 15, 1932.

[12] V. Belevitch, "Tchebyshev filters and amplifiers networks," *Wireless Eng.*, vol. 29, pp. 106–107; April, 1952.

H. J. Orchard, "Formulae for ladder filters," *Wireless Eng.*, vol. 30, pp. 3–5; January, 1953.

inversely with the square root of its length measured in half wavelengths. The change in (19)–(22) required to give the same pass-band characteristic when the length of a given transmission line section is increased to $n$ half wavelengths is readily seen from (11). Here, if the corresponding $p$ is replaced by $np$, no change in $Y_0$ results when the adjacent inversion factors are divided by $n$.

This section is concluded by establishing the exact equivalence between a direct-coupled filter and a quarter-wave-coupled filter (Fig. 3) employing Mumford's approximation for a quarter wavelength of waveguide.

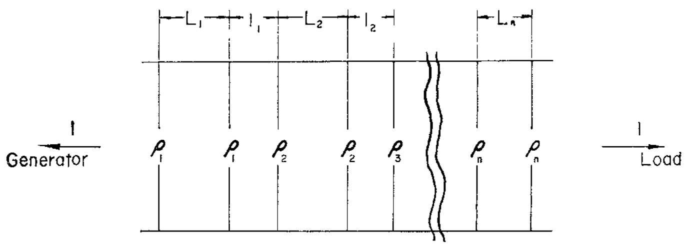



Fig. 3—Schematic of a quarter-wave-coupled filter.

First, the admittance transformation,

$$\begin{pmatrix} kp & 1+(kp)^2 \\ 1 & kp \end{pmatrix}, \tag{14}$$

for $k=\frac{1}{2}$ (quarter-wave coupling), 3/2 (three-quarter-wave coupling), etc., is that used by Mumford[8] to approximate sections of waveguide an odd number of quarter wavelengths long. We are required to show that a matrix product of the form,

$$\begin{pmatrix} 1 & (f_1-k)p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} kp & 1+(kp)^2 \\ 1 & kp \end{pmatrix}\begin{pmatrix} 1 & (f_2-2k)p \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & (f_n-k)p \\ 0 & 1 \end{pmatrix}, \tag{15}$$

is precisely equivalent to (7), since we have already seen how in (4), the matrices involving the $f_i$'s are the admittance transformations of resonant cavities, except for a multiplicative factor. This is done by replacing the first and last terms with the help of the identities,

$$\begin{pmatrix} 0 & f_i \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}\begin{pmatrix} 0 & f_i \\ 1 & -kp \end{pmatrix} = f_i\begin{pmatrix} 1 & (f_i-k)p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -kp & f_i \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}\begin{pmatrix} 0 & f_i \\ 1 & 0 \end{pmatrix}, \tag{16}$$

and the other terms involving $f$'s by using the identity,

$$\begin{pmatrix} -kp & f_i \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}\begin{pmatrix} 0 & f_i \\ 1 & -kp \end{pmatrix} = f_i\begin{pmatrix} 1 & (f_i-2k)p \\ 0 & 1 \end{pmatrix}. \tag{17}$$

Then the matrices involving $1+(kp)^2$ are eliminated using the identity,

$$\begin{pmatrix} 0 & f_i f_{i+1} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & f_i \\ 1 & -kp \end{pmatrix}\begin{pmatrix} kp & 1+(kp)^2 \\ 1 & kp \end{pmatrix}\begin{pmatrix} -kp & f_{i+1} \\ 1 & 0 \end{pmatrix}. \tag{18}$$

This equivalence, of course, only operates in one direction. For a direct-coupled filter network there is no equivalent quarter-wave-coupled filter network, in general, without specifying the termination.

## Design Formulas

When expressions for $f_i$,[11,12] are substituted in (8) and suitable allowance is made, according to (13), for a change in frequency scale, general formulas are obtained for the VSWR's of the reflecting elements of maximally flat or equal ripple, direct-coupled filters. The equivalence between (15) and (7), results in similar formulas for the VSWR's of the reflecting elements of quarter-wave-coupled filters. Table I gives these formulas together with the figures, equations, and definitions required for design purposes.

Table II gives formulas for the line lengths separating the reflecting elements, together with an equation and the definitions required for design. These formulas are conventional except for the allowance for end effects. This addition has been made because the author's experience shows that in some applications end effect correction must be taken into account in order to avoid significant experimental error.

The use of Table I formulas requires the determination of $h$ (for equal-ripple performance), $n$, $\omega_0$, and $t$ in that order. The value of $h$ is readily obtained from the tolerance given for the pass-band ripple or VSWR. Selection of $n$ is made to yield the desired skirt steepness. The formulas for $P_L$ are used for this purpose where $\omega$ equals a frequency scale factor multiplied by the departure of the frequency from the filter midband. For quick estimates this can be done in terms of frequency but for greater accuracy the use of guide wavelengths is recommended. Once $n$ and $h$ are fixed, $\omega_0$ is found from the corresponding expression for $P_L$. For example, if a two-section maximally flat filter is specified to have 10-db loss at frequencies corresponding to $\lambda_{g_1}$ and $\lambda_{g_2}$, then $P_L = 10 = 1+\omega_0^4$, so that $\omega_0 = \sqrt[4]{3}$. Where the bandwidth of a maximally flat filter is specified at the 3-db points $\omega_0 = 1$; if the bandwidth of an equal ripple filter is specified at the extremes of the equal ripple tolerance, $\omega_0 = 1$. The $t$ occurring in the formulas is a generalization of the notion of total $Q$ introduced by Mumford. It is determined by the requirement that $t \sin \theta = \omega_0$ for the two specified guide wavelengths $\lambda_{g_1}$, and $\lambda_{g_2}$ at which symmetrical behavior is expected.

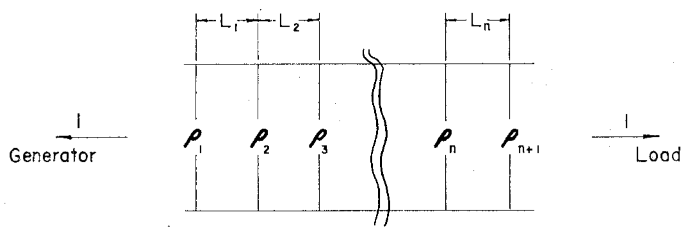
## Definitions for Table I

The VSWR of the reflecting element is $\rho$. $P_L$ is the insertion loss function obtained by dividing the avail-

TABLE I

INVERSION FACTOR FORMULAS

| | Equal Ripple | Maximally Flat |
|---|---|---|
| Direct-Coupled Filter | $\rho_1 = t\cdot\dfrac{2\sin\frac{\pi}{2n}}{\gamma}$<br>$\rho_2 = t^2\cdot\dfrac{4\sin\frac{3\pi}{2n}\cdot\sin\frac{\pi}{2n}}{\gamma^2+\sin^2\frac{\pi}{n}}$<br>⋮<br>$\rho_i = t^2\cdot\dfrac{4\sin\frac{(2i-1)\pi}{2n}\cdot\sin\frac{(2i-3)\pi}{2n}}{\gamma^2+\sin^2\frac{(i-1)\pi}{n}}$<br>⋮<br>$\rho_{n+1} = \rho_1$ (19) | $\rho_1 = t\cdot 2\sin\frac{\pi}{2n}$<br>$\rho_2 = t^2\cdot 4\sin\frac{3\pi}{2n}\cdot\sin\frac{\pi}{2n}$<br>⋮<br>$\rho_i = t^2\cdot 4\sin\frac{(2i-1)\pi}{2n}\cdot\sin\frac{(2i-3)\pi}{2n}$<br>⋮<br>$\rho_{n+1} = \rho_1$ (20) |




| | Equal Ripple | Maximally Flat |
|---|---|---|
| Quarter-Wave-Coupled Filter | $\rho_1 = t\cdot\dfrac{2\sin\frac{\pi}{2n}}{\gamma} - k$<br>$\rho_2 = t\cdot\dfrac{2\sin\frac{3\pi}{2n}\cdot\gamma}{\gamma^2+\sin^2\frac{\pi}{n}} - 2k$<br>⋮<br>$= t\cdot\dfrac{2\sin\frac{(2i-1)\pi}{2n}\cdot\left(\gamma^2+\sin^2\frac{\pi}{n}\right)\cdots}{\gamma\cdot\left(\gamma^2+\sin^2\frac{2\pi}{n}\right)\cdots} - 2k$<br>⋮<br>$\rho_n = \rho_1$<br>($n$ odd only) (21) | $\rho_1 = t\cdot 2\sin\frac{\pi}{2n} - k$<br>$\rho_2 = t\cdot 2\sin\frac{3\pi}{2n} - 2k$<br>⋮<br>$\rho_i = t\cdot 2\sin\frac{(2i-1)\pi}{2n} - 2k$<br>⋮<br>$\rho_n = \rho_1$ (22) |

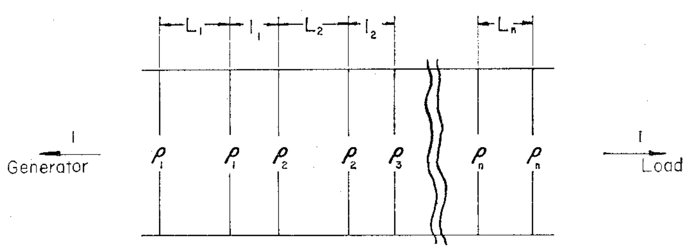



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TABLE I (*Cont'd*)

| | Defining Equations | |
|---|---|---|
| | $P_L = 1 + [hT_n(\omega)]^2$ <br> $\gamma = \sinh \frac{1}{n} \sinh^{-1} \frac{1}{h}$ (23) | $P_L = 1 + \omega^{2n}$ |
| | $t = \omega_0 \csc \frac{\pi(\lambda_{g_1} - \lambda_{g_2})}{\lambda_{g_1} + \lambda_{g_2}}$ (24) | |

TABLE II

LINE LENGTH FORMULAS

$$L_i = \frac{\lambda_{g_1}\cdot\lambda_{g_2}}{\lambda_{g_1}+\lambda_{g_2}} - \left(\frac{\phi_i}{2} + \frac{\phi_{i+\epsilon}}{2}\right) - T_i - T_{i+\epsilon}$$

$$\doteq \frac{\lambda_{g_1}+\lambda_{g_2}}{4} - \left(\frac{\phi_i}{2} + \frac{\phi_{i+\epsilon}}{2}\right) - T_i - T_{i+\epsilon}. \tag{25}$$

$$l_i = k\cdot\frac{\lambda_{g_1}\cdot\lambda_{g_2}}{\lambda_{g_1}+\lambda_{g_2}} - \left(\frac{\phi_i}{2} + \frac{\phi_{i+1}}{2}\right) - T_i - T_{i+1}.$$

DEFINING EQUATION

$$\rho_i = \cot^2 \frac{\phi_i}{2}. \tag{26}$$

able power by the power into the load. The number of resonant sections is $n$. The pass-band, insertion-loss tolerance is given by $h$ and is related to the maximum, pass-band VSWR, $\rho_{\max}$, by

$$h = (\rho_{\max} - 1)/(2\sqrt{\rho_{\max}}).$$

In the expression for $t$, $\lambda_{g_1}$ and $\lambda_{g_2}$ are the guide wavelengths at which the skirt insertion loss is specified and $\omega_0$ is the corresponding value of $\omega$ determined from $P_L$. $k = \frac{1}{2}$ (quarter-wave coupling), and 3/2 (three-quarter wave coupling), etc. $\omega$ is equal to $t\cdot\sin(\pi\lambda_{g_0}/\lambda_g)$, where $\lambda_{g_0}$ is the mean guide wavelength given by $2\lambda_{g_1}\cdot\lambda_{g_2}/(\lambda_{g_1}+\lambda_{g_2}) \doteq (\lambda_{g_1}+\lambda_{g_2})/2$.

DEFINITIONS FOR TABLE II

$\epsilon$ is unity for direct-coupled filters and zero for quarter-wave-coupled filters. $T_i$ is the end effect error associated with the $i$th reflecting element. $\phi$ is positive for series capacities and shunt inductances, and negative for series inductances and shunt capacities.

## COMPARISON WITH PREVIOUS RESULTS

*Lawson and Fano*[1]

Neglecting $X_S$ and $X_L$, comparison of their (146) with (11) of this paper indicates that $\rho_i = X_i^{-2}$. We note that (8) can be written, with a suitable frequency transformation

$$\rho_1 = tC_1,\ \rho_2 = t^2C_1L_2,\ \rho_3 = t^2L_2C_3,\ \cdots,\ \rho_{n+1} = t\cdot(C_n \text{ or } L_n). \tag{27}$$

Since our $t$ is defined to be the reciprocal value of $\sin\theta$ at the band edges, and $p$ is equivalent to $Z$ in their (145), we have $t = 1/Lw$. Hence their (156) to (159) are seen to be equivalent to (26), except for the subscript on $L$ which should be a superscript. Consequently (19) and (20) are essentially special cases of general formulas given in their paper.

Eqs. (160) to (163)[1] can also be obtained. According to Lawson and Fano, $w$ is the frequency bandwidth for unity value of $\omega'$ so that our value of $\omega_0 = 1$. Thus by (24),

$$t = \csc\frac{\pi(\lambda_{g_1} - \lambda_{g_2})}{\lambda_{g_1}+\lambda_{g_2}} \doteq \frac{\lambda_{g_1}+\lambda_{g_2}}{\pi(\lambda_{g_1} - \lambda_{g_2})} \doteq \frac{2\lambda_{g_0}}{\pi\Delta\lambda_g}, \tag{28}$$

where $\Delta\lambda_g = \lambda_{g_1} - \lambda_{g_2}$ and $\lambda_{g_0}$ is the mean guide wavelength. Now

$$\Delta\lambda_g \doteq \frac{\lambda_{g_0}^3}{\lambda_0^3}\Delta\lambda \doteq \frac{\lambda_{g_0}^3 w}{\lambda_0^2\omega_0}$$

so that

$$t \doteq \frac{2\,\lambda_0^2\,\omega_0}{\lambda_{g0}^2\cdot w}, \tag{29}$$

and finally since $(\lambda_0/\lambda_{g_0})^2 = 1 - (\omega_c/\omega_0)^2$ we have

$$t \doteq \frac{2}{\pi}\frac{\omega_0}{w}\left\{1 - \left(\frac{\omega_c}{\omega_0}\right)^2\right\}. \tag{30}$$

Since

$$|b_i| = \rho_i^{1/2} - \rho_i^{-1/2} \doteq \rho_i^{1/2}, \tag{31}$$

we obtain their (160) except that $\omega$ should be $w$. In their (161) and (162), $C_1$ should be taken outside of the radical sign.

*Southworth*[2]

If we put $C_1 = f_1 = 2\sin\pi/2n$ in (160) of Lawson and Fano,[1] we immediately obtain (9.2-7) of Southworth. His (9.2-8) is immediately obtained from (20), if we use the above approximation for $t$, put $|\overline{B}_m| = \sqrt{\rho_m}$ and replace the product of sines by the difference in cosines.

*Mumford*[3]

It is clear from (A10) of Mumford[3] and (27) of this paper that in the narrow-band limit, $Q_t$ is approximately equal to $\pi t/2$ when $\omega_0$ is chosen to be unity. Now the $f_i$'s that occur in (6) when multiplied by $\pi/4$

are precisely the $Q_i$'s in his (15) in the small angle limit. Thus (22) can be written:

$$Q_1 = \pi/4\rho_1 = \frac{\pi}{4}\cdot\left\{\frac{4}{\pi}Q_t \sin\frac{\pi}{2n} - k\pi/4\right\}$$

$$Q_i = \pi/4\rho_i = \frac{\pi}{4}\cdot\left\{\frac{4}{\pi}Q_t \sin\frac{(2i-1)\pi}{2n} - \frac{k\pi}{2}\right\}$$

$$\vdots$$

$$Q_r = Q_1.$$

These are precisely Mumford's results when the selectivities of the coupling lines are included.

If assumption 1 is satisfied by the reflecting elements, then the above remarks concerning Mumford's paper are true. This condition, though sufficient for a synthesis of quarter-wave-coupled filters, in terms of a ladder network prototype is not necessary. As explained in Appendix I, it is possible to define the $Q$ of a cavity terminated in more general reflecting elements. For such cases, the characteristics of the reflecting elements can be determined from the required $Q$'s by means of formulas derived for each case. For the cases of usual interest, three such formulas have been given in Riblet and Reed.[13] There it was pointed out by Reed that the formula relating to the case of constant susceptance is equivalent to Mumford's. It is also indicated that the formula for the inductive case is equivalent to that given by Reed.[14]

Eq. (22) gives the value of $\rho_i$ directly since it can be shown for the inductive case, that the error in approximating $Q_i$ by $\pi/4\rho_i$ is of the order of $\phi_0^3/5$.

*Cohn*[4]

The value of $L$ in Fig. 5 of Cohn[4] is the same as the $t^{-1}$ used in this paper except for a typographical ambiguity and the use of the small angle approximation. The values of $X_{i,i+1}$, except possibly for sign, follow from

$$|X_{i,i+1}| = |b_i|^{-1} = \frac{(\rho_i)^{-1/2}}{1-(\rho_i)^{-1}}.$$

For completeness, the frequency transformation due to Cohn[4] is derived at this point. In the limit of large susceptances, $\rho_i = b_i^2$. From (13), we see that a frequency variation common to the inversion factors can be included in the frequency variable. Now, if $\rho_i = \bar{\rho}_i f^2(p)$ with $\bar{\rho}_i$ constant, the theory is applicable when we replace $p$ by $f(p)\cdot p$ except for a relatively small error in the end elements. For inductances which vary directly as $\lambda_g$ (this is approximated by waveguide irises), $f(p) = \lambda_g/\lambda_{g_0}$, where $\lambda_{g_0}$ is the midband guide wavelength. Then

$$p = j\lambda_g/\lambda_{g_0}\sin(\pi\lambda_{g_0}/\lambda_g). \tag{32}$$

Using the small angle approximation for $\sin\theta$, we have

$$p \doteq j\pi(\lambda_g/\lambda_{g_0} - 1),$$

with $\lambda_{g_0} = (\lambda_{g_1}+\lambda_{g_2})/2$, where $\lambda_{g_1}$ and $\lambda_{g_2}$ are the guide wavelengths at the band edges.

This frequency variable has, as Cohn has pointed out, the important property that the response curve of the filter is symmetrical in $\lambda_g$ rather than in $1/\lambda_g$. Curiously, (24) for $t$ is not altered by this transformation, a statement which is not true for capacitive irises. The principal effect of this transformation on the design procedure is to alter the midband guide wavelength which enters into the determination of $L$ and $l$ in (25). For bandwidths as great as 10 per cent, and difference between the two formulas for $L$, due to this transformation, is generally less than the error due to the neglect of $T_i$, and so we have given the formula for the mean guide wavelength which is a rigorous consequence of (1) and (2). The approximate formula is simpler for computation and is to be recommended accordingly for design purposes whenever inductive irises are used.

*Riblet*[5]

Riblet[5] is concerned with a first order equivalence between direct-coupled filters and quarter-wave-coupled filters in contrast to the zero order equivalence established in (14)–(17). Its interest lies in the fact that a synthesis in terms of quarter-wave-coupled cavities requires less restrictive assumptions on the nature of the reflecting elements (see Appendix I) than is required for a direct-coupled filter synthesis, based on a ladder network prototype. Accordingly, the procedure is capable of a high degree of accuracy and generality when used with the recommended frequency transformation.

## Comments on Assumption 1

The applicability of this assumption to practical waveguide reflecting elements must be justified by consideration of their properties or by the construction of experimental filters. Admittance transformations for shunt susceptances have been given[13] which are constant and which vary inversely and directly with $\lambda_g$. These are correct to the first order in a frequency variable, $\Omega$, where $\Omega \doteq jp/\pi$. Written in the form of (1), these become

$$\begin{pmatrix} \dfrac{k_2\Omega}{\sin\phi_0} & j\sqrt{\dfrac{1+\cos\phi_0}{1-\cos\phi_0}}\left(1-\dfrac{k_1\Omega}{\sin\phi_0}\right) \\ j\sqrt{\dfrac{1-\cos\phi_0}{1+\cos\phi_0}}\cdot\left(1+\dfrac{k_1\Omega}{\sin\phi_0}\right) & \dfrac{k_2\Omega}{\sin\phi_0} \end{pmatrix}, \tag{33}$$

where

$$k_1 = \pm\sin^2\phi_0\cos\phi_0$$

$$k_2 = \phi_0 \pm \sin\phi_0\cdot\cos\phi_0,$$

[13] J. Reed and H. J. Riblet, "Discussion on synthesis of narrow-band direct-coupled filters," Proc. IRE, vol. 41, pp. 1058–1059; August, 1953. See (4).

[14] J. Reed, "Low Q microwave filters," Proc. IRE, vol. 38, pp. 793–796; July, 1950. See p. 794.

and the upper signs are used with inductances, the lower signs are used with capacities, and they are replaced by zero for constant susceptances. Of course $\phi_0$ is the value of $\phi$, previously defined, when $\Omega=0$. For inductances, the terms on the principal diagonal tend to zero with increasing susceptance. Accordingly the theory is particularly applicable to this case, as Cohn[4] has pointed out.

## Comments on Assumption 2

The principal justification is the following theorem: The matrix product,

$$\cos^n\theta\begin{pmatrix}0 & c_1\sqrt{t}\\ \dfrac{1}{c_1\sqrt{t}} & 0\end{pmatrix}\begin{pmatrix}1 & \bar{p}\\ p & 1\end{pmatrix}\begin{pmatrix}0 & c_2t\\ \dfrac{1}{c_2t} & 0\end{pmatrix}\begin{pmatrix}1 & \bar{p}\\ p & 1\end{pmatrix}$$
$$\cdots\begin{pmatrix}0 & c_nt\\ \dfrac{1}{c_nt} & 0\end{pmatrix}\begin{pmatrix}1 & \bar{p}\\ p & 1\end{pmatrix}\begin{pmatrix}0 & c_{n+1}\sqrt{t}\\ \dfrac{1}{c_{n+1}\sqrt{t}} & 0\end{pmatrix}, \quad (34)$$

has the form

$$\cos^n\theta\begin{pmatrix}D(pt,\bar{p}/t) & C(pt,\bar{p}/t)\\ B(pt,\bar{p}/t) & A(pt,\bar{p}/t)\end{pmatrix}, \quad (35)$$

where $A$, $B$, $C$, and $D$ are polynomials in $pt$ and $\bar{p}/t$, and the only $n$th degree terms are $(pt)^n$ and $(\bar{p}/t)^n$. That is, terms of the form $p^k\bar{p}^{n-k}$ do not occur.

When $\bar{p}=p=j\tan\theta$, the matrix product of the theorem represents the exact admittance transformation of a cascade of admittance inverters, equally spaced on a uniform transmission line, whose dependence on $t$ is consistent with the form obtained in (13) from a ladder network prototype. Moreover the determinant of (35) is unchanged by replacing $\bar{p}$ by zero, and $\cos\theta$ by one.

Comparison of the input impedance and insertion loss function obtained from this matrix product, with the corresponding functions obtained with $\bar{p}$ put equal to zero, will indicate the error in the approximate solution. Now the coefficient of every power of $p$ in $A$, $B$, $C$, and $D$ will contain terms contributed by $p$ and by $\bar{p}$ since $\bar{p}$ must be replaced by $p$ in an exact calculation. Although the $n$th degree term contains only $p^n$ and $\bar{p}^n$, in general, terms of the form $p^r\bar{p}^s$ appear. Now the theorem states that in each coefficient, the contribution of $p^r$ is of the order of $t^r$ while the corresponding contribution of $\bar{p}^r$ is of the order of $t^{-r}$. In general then, the error in neglecting $\bar{p}$ terms is of the order of $1/t^2$, and for the coefficient of $p^n$, the error is of the order of $1/t^{2n}$. The limit for large $t$, with $tp$ fixed, can be attained then by putting $\bar{p}=0$ and $\cos\theta=1$.

Since the determinant of (34) is unity, the insertion loss function is given, in the limit, by

$$1/4\cos^{2n}(\theta)\,|\,A(pt)+B(pt)+C(pt)+D(pt)\,|^2. \quad (36)$$

This is an even polynomial in $\tan\theta$ of the form,

$$a_{2n}t^{2n}\tan^{2n}\theta+a_{2n-2}t^{n-2}\tan^{2n-2}\theta+\cdots.$$

Multiplication by $\cos^{2n}(\theta)$ gives the expression,

$$a_{2n}t^{2n}\sin^{2n}\theta+a_{2n-2}t^{2n-2}\sin^{2n-2}(\theta)\cos^2(\theta)+\cdots.$$

Now the resulting terms in $\cos^2\theta$ can be replaced by $\sin^2\theta$ but the error in replacing $\cos^2\theta$ by 1 is always of the order of $1/t^2$. Thus in the exact admittance transformation (24) approaches, as $t\to\infty$, the limit obtained from it by replacing $\bar{p}$ by zero and $p$ by $j\sin\theta$. Assumption 2 is thus rigorously justified.

It may be of interest to observe that assumption 2 is not essential for a synthesis based on a prescribed insertion loss function. The exact transmission line synthesis procedure for filters consisting only of equal length impedance transformers mentioned by Riblet[15] and discussed in detail by Seidel[16] is immediately applicable so long as assumption 1 is valid. Seidel has arrived at the same conclusion and has applied this theory to the design of direct-coupled filters. He is concerned, however, with transmission line elements which are nominally a quarter wavelength long so that his results do not appear to be directly applicable to the narrow band problem. This paper is the direct consequence of early efforts to carry through the exact synthesis of a narrow band filter on the basis of the exact procedure. When it was discovered that the $p$ roots[15] had to be calculated with extreme accuracy, in order to satisfy (2) for physical realizability, the present analysis of the approximate solution was forced on the writer.

## Conclusions

Formulas previously given[1,2,4] are rederived on the basis of two approximations which are shown, under general conditions, to be necessary and sufficient for a general synthesis of direct-coupled filters having the frequency behavior associated with ladder network prototypes. Differences in the formulas, not to due to typographical errors, are traced primarily to an approximation used to express the susceptance of a reflecting element in terms of its VSWR, and secondarily to small angle approximations and the use of differentials in place of differences. Formulas[3] for the design of quarter-wave-coupled filters are rederived on the basis of an exact equivalence which is given for a direct-coupled filter and a "Mumford" quarter-wave-coupled filter.

The usefulness of the formulas is extended by showing how the VSWR's of the reflecting elements are altered when the interconnecting lines are lengthened in multiples of half of a guide wavelength. It is shown how other approximate solutions[1,2,4] are a rigorous limit of an exact solution and that the error in each coefficient of the insertion loss function, made by replacing each half

[15] H. J. Riblet, "General synthesis of quarter-wave impedance transformers," IRE Trans. on Microwave Theory and Techniques, vol. MTT-5, pp. 36–43; January, 1957. See p. 38.

[16] H. Seidel, "Synthesis of a class of microwave filters," IRE Trans. on Microwave Theory and Techniques, vol. MTT-5, pp. 107–114; April, 1957.

wavelength of transmission line by a series resonant element, is of the order of one half the square of the percentage bandwidth measured in guide wavelengths.

## Appendix I

We are concerned here with the conditions satisfied by general reflecting elements if it is required that they are to be used in a general synthesis of direct-coupled filters[17] based on a ladder network prototype. The proofs assume that the primary frequency sensitive elements of the direct-coupled filters consist of *equal* lengths of transmission line which separate them or, as we shall see, series resonant elements which approximate transmission lines in their frequency behavior. Although the assumption of equal line lengths may not be a physical necessity, it is difficult to imagine any general synthesis theory based on a ladder network prototype which does not require this simplifying assumption.

Consider the admittance transformation, $(R)$, ($D$, $C$, $B$, $A$ matrix), of a symmetric, nonresonant, reflecting element, written,

$$(R) = \begin{pmatrix} a_1\theta + a_2\theta^2 + a_3\theta^3 + \cdots & j(c_0 + c_1\theta + c_2\theta^2 + c_3\theta^3 + \cdots) \\ j(b_0 + b_1\theta + b_2\theta^2 + b_3\theta^3 + \cdots) & a_1\theta + a_2\theta^2 + a_3\theta^3 + \cdots \end{pmatrix}$$

$$(R) = \begin{pmatrix} A(\theta) & C(\theta) \\ B(\theta) & A(\theta) \end{pmatrix},$$

where $A$, $B$, $C$ satisfy the following conditions:

1) $A$, $B$, and $C$ are analytic functions of the complex variable $\theta$ in the vicinity of $\theta = 0$.
2) $A^2 - BC = 1$

If we denote by $(T)$, the admittance transformation of a uniform transmission line so that

$$(T) = \begin{pmatrix} \cos(\theta + \pi) & j\sin(\theta + \pi) \\ j\sin(\theta + \pi) & \cos(\theta + \pi) \end{pmatrix},$$

then we may prove the following theorem.

*Theorem A*

If $\omega$ is a frequency variable assumed to be an odd function of $\theta$ then matrix products of the form,

$$(R_1)(T)(R_2)(T) \cdots (T)(R_{n+1}),$$

cannot yield a general representation of the input impedance functions in $\omega$ of ladder network prototypes.

It should be observed that the implied assumption has been made that resonance, $\omega = 0$, occurs when the connecting lines are all one half-wavelength long. This assumption is no restriction, since identical line lengths can be added to or subtracted from each $(R_i)$ without changing the conditions of the theorem or essentially altering the input admittance function of the cascade. Moreover, it is the assumption previously made.[1,2,4,5]

*Proof*

We require that for arbitrary $(R)$ satisfying conditions 1 and 2 that

$$(R)(T)(R) = \begin{pmatrix} 1 & 4Q\omega \\ 0 & 1 \end{pmatrix},$$

since the right-hand matrix represents the ladder network element consisting of a single resonant element of given $Q$. This requirement is certainly satisfied by the reflecting elements mentioned previously.[1-5]

If

$$(T) = \begin{pmatrix} -(1 - \theta^2/2 \cdots) & -j(\theta - \theta^3/6 \cdots) \\ -j(\theta - \theta^3/6 \cdots) & -(1 - \theta^2/2 \cdots) \end{pmatrix}$$

and put $\omega = \theta - \omega_3\theta^3$, the following conditions on the coefficients of $(R)$ are found.

$$\begin{aligned} a_1 &= \phi/2 & b_0 &= \phi & c_0 &= 1/\phi \\ a_2 &= 0 & b_1 &= 0 & c_1 &= 0 \\ a_3 &= -\phi^3/48 + (\phi - \phi^3)\omega_3 & b_2 &= -\phi^3/24 - \phi/12 + 2(\phi - \phi^3)\omega_3 & c_2 &= -\frac{5\phi}{24} + \frac{1}{12\phi} - 2\left(\frac{1}{\phi} - \phi\right)\omega_3, \end{aligned} \quad (37)$$

where $\phi$ is determined by

$$4Q = 1/\phi^2 - 1.$$

It is interesting that for a suitable choice of the frequency behavior of the reflecting elements, a shunt resonant element can be exactly obtained with a half wavelength of transmission line. It should be recalled, of course, that the $Q$ previously used[3,5,14] is only approximate and assumes that the resonant element is terminated in a match.

The theorem follows, however, when an attempt is made to combine three reflecting elements satisfying (37). In this case, it is readily shown that a product of the form $(R_1)(T)(R_2)(T)(R_3)$ will be, in general, of second degree in $\omega$ in both the numerator and the denominator of the input admittance function. This, of course, makes it impossible to represent the general two-element ladder network in the required form.

[17] This term is defined by Fig. 2 and has also been previously illustrated.[1,2,4,5]

We may now replace $(T)$ by

$$(T) = \begin{pmatrix} -1 & 0 \\ j \sin(\theta + \pi) & -1 \end{pmatrix},$$

which is an approximation justified in Appendix II and prove the following theorem.

*Theorem B*

If $\omega$, the frequency variable, vanishes for $\theta = 0$ (half wavelength spacing) and $\sin \theta$ is an analytic function of $\omega$, then matrix products of the form,

$$(R_1)(\tilde{T})(R_2)(\tilde{T}) \cdots (\tilde{T})(R_{n+1}),$$

can define the general input admittance function of all ladder network prototypes only if

$$(R_i) = \begin{bmatrix} 0 & j\sqrt{\rho_i} \\ \dfrac{j}{\sqrt{\rho_i}} & 0 \end{bmatrix},$$

where $\rho_i$ is a positive real constant.

*Proof*

We consider consequences of the assumption that

$$\begin{pmatrix} A & C \\ B & A \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -j \sin\theta & -1 \end{pmatrix} \begin{pmatrix} A & C \\ B & A \end{pmatrix} = \begin{pmatrix} 1 & jf\omega \\ 0 & 0 \end{pmatrix},$$

where $f$ is a constant. If we put $\omega = 0$, it is readily shown that $A(0) = 0$, while $B(0) \cdot C(0) = -1$. Moreover, since

$$\begin{bmatrix} c_1 p\sqrt{t} & c_1\sqrt{t} \\ \dfrac{1}{c_1\sqrt{t}} & \dfrac{\bar{p}}{c_1\sqrt{t}} \end{bmatrix} \begin{bmatrix} D(pt, \bar{p}/t) & tC(pt, \bar{b}/t) \\ 1/tB\ pt, \bar{p}/t) & A(pt, p/t) \end{bmatrix} \begin{bmatrix} 0 & c_{n+1}\sqrt{t} \\ \dfrac{1}{c_{n+1}\sqrt{t}} & 0 \end{bmatrix}. \tag{39}$$

$2AB + jA^2 \sin \theta \equiv 0$, from the lower left-hand element of the product, we conclude that $A \equiv 0$ or $jA \sin \theta = -2B$. The latter case is excluded by the fact that $B(0) \neq 0$. From the upper left-hand element of the product, we conclude that $CB \equiv -1$ and from the upper right-hand element that $-C^2 \sin \theta = f\omega$. Therefore, the permitted form for $(R_i)$ is

$$(R_i) = \begin{bmatrix} 0 & j\sqrt{\dfrac{f\omega}{\sin\theta}} \\ \sqrt{\dfrac{\sin\theta}{f\omega}} & 0 \end{bmatrix},$$

showing that some frequency variations in $(R_i)$ is permitted. However, if one considers a cascade containing three of these reflecting elements, then the condition $\omega = \sin \theta$ can be shown to be necessary in order that the frequency behavior of two element ladder network prototypes be representable in the given frequency variable, $\omega$.

## Appendix II

The proof of the theorem underlying assumption 2 follows by mathematical induction.

Consider

$$\begin{bmatrix} 0 & c_n t \\ \dfrac{1}{c_n t} & 0 \end{bmatrix} \begin{bmatrix} 1 & \bar{p} \\ p & 1 \end{bmatrix} = \begin{bmatrix} c_n pt & c_n t \\ \dfrac{1}{c_n t} & 1/c_n \bar{p}/t \end{bmatrix}. \tag{38}$$

It has the general form,

$$\begin{bmatrix} D(pt, \bar{p}/t) & tC(pt, \bar{p}/t) \\ \dfrac{1}{t} B(pt, \bar{p}/t) & A(pt, \bar{p}/t) \end{bmatrix}, \tag{39}$$

in which the highest power of $p$ occurs in $D$, and the highest power of $\bar{p}$ occurs in $A$. Furthermore the form of (39) is not changed by multiplication by matrices of the form of (38), while the highest power of $p$ still occurs in $D$ and the highest power of $\bar{p}$ occurs in $A$. Thus the original product (34) can be written

When this multiplication is carried through, one obtains

$$\begin{bmatrix} \dfrac{c_1}{c_{n+1}}(ptC + A) & c_1 c_{n+1}(ptD + B) \\ \dfrac{1}{c_1 c_{n+1}}(C + \bar{p}/tA) & \dfrac{c_{n+1}}{c_1}(D + \bar{p}/tB) \end{bmatrix}.$$

This is the result claimed in the theorem. The $C$ term in this matrix contains $(pt)^n$ and the $B$ term contains $(\bar{p}/t)^n$. No cross product terms having a total degree of $n$ can occur.